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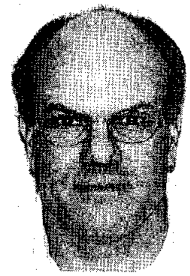

Riaan Booysen

Editorial Note

We have another tutorial presentation on the Field Equivalence Principle, this time involving a fictitious cylindrical surface. Dr. Riaan Booysen shows interesting field plots, obtained from computed numerical results. The author is with Grintek Antennas in the Republic of South Africa.

Please send me your comments on this or the previous tutorial, which appeared in the August issue. If you have an interesting tutorial article, or any other item for contribution to this column, please contact me.

In this issue of the *Magazine* you will find a call for applications for our annual IEEE AP-S Undergraduate Scholarships. The application deadline is **April 30, 2001**. Please bring it to the attention of any students you know.

An Illustrative Equivalence Theorem Example

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Keywords: Electromagnetic fields; electromagnetic analysis; electromagnetic theory; equivalence principle; electromagnetic engineering education; electrical engineering education; cylinders

Students often struggle with the Equivalence Theorem [1-3] as a concept. The example presented here might help them understand how it can be implemented in different forms.

Consider a TM_z plane wave, propagating in the $+x$ direction in free space, as shown in Figure 1. There are no scattering bodies present. A fictitious infinite cylinder, of radius R, centered at the origin, has equivalent electric and magnetic currents $\overline{\mathbf{J}}_{s1}$ and $\overline{\mathbf{M}}_{s1}$, respectively, placed on its surface, S. These currents are defined in Equations (1) and (2) and, together with the impressed field, they maintain the true (incident) field external to S. The electric- and magnetic-field expressions are given by Equations (3) and (4). The reason for selecting this example is that the true fields are known exactly, whereas this would not be the case if a real scatterer were present.

$$\overline{\mathbf{J}}_{s1} = \hat{\mathbf{n}}_1 \times \overline{\mathbf{H}}_1, \tag{1}$$

$$\overline{\mathbf{M}}_{s1} = -\hat{\mathbf{n}}_1 \times \overline{\mathbf{E}}_1. \tag{2}$$

The vector $\hat{\mathbf{n}}_1$ is a unit vector, normal to the surface S, pointing towards region 1. $\overline{\mathbf{E}}_1$ and $\overline{\mathbf{H}}_1$ are the total electric and magnetic fields, respectively, exterior to S. The scattered fields produced by the equivalent surface currents are obtained in terms of the Hankel functions, as shown in Equations (3) and (4), below [1].

An Illustrative Equivalence Theorem Example*

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Consider a TM_z plane wave, propagating in the +*x* direction in free space, as shown in Figure 1. There are no scattering bodies present. A fictitious infinite cylinder, of radius *R*, centered at the origin, has equivalent currents $\overline{\mathbf{J}}_{s1}$ and $\overline{\mathbf{M}}_{s1}$, respectively, placed on its surface *S*. These currents are defined in Equations (1) and (2) and, together with the impressed field, they maintain the true (incident) field external to *S*. The electric- and magnetic-field expressions are given by Equations (3) and (4). The reason for selecting this example is that the true fields are known exactly, whereas this would not be the case if a real scatterer were present.

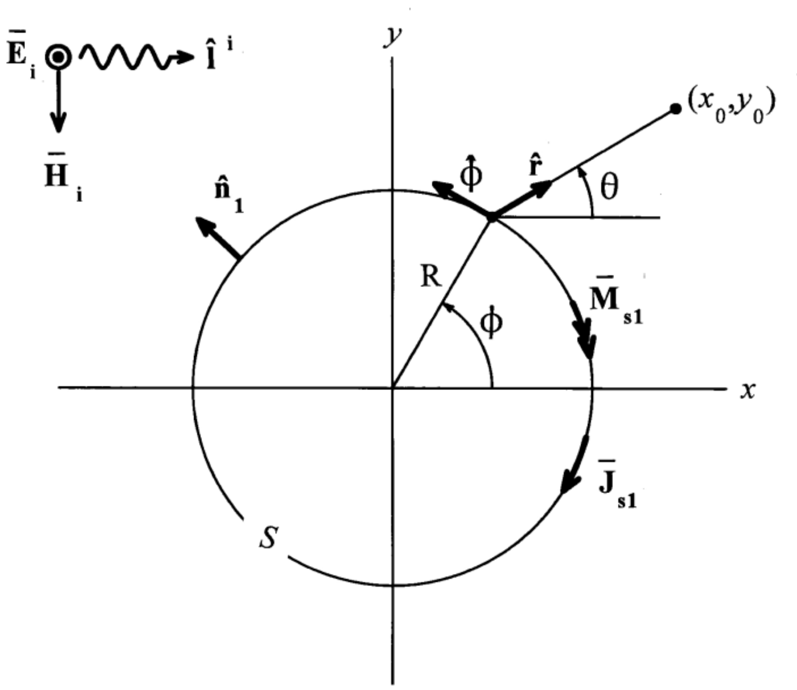

Figure 1. Equivalence theorem example – fictitious surface currents producing incident field

$$\overline{\mathbf{J}}_{s1} = \hat{\mathbf{n}}_1 \times \overline{\mathbf{H}}_1 \tag{1}$$

$$\overline{\mathbf{M}}_{s1} = -\hat{\mathbf{n}}_1 \times \overline{\mathbf{E}}_1 \tag{2}$$

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$$\overline{\mathbf{E}}^s(\overline{\mathbf{J}}_s,\overline{\mathbf{M}}_s) = -\frac{\omega\mu}{4}\int_c \overline{\mathbf{J}}_s H_0^{(2)}(kr)dC + \frac{k}{4\omega\varepsilon}\int_c (\nabla_s \bullet \overline{\mathbf{J}}_s)\,\hat{\mathbf{r}}\, H_1^{(2)}(kr)dC$$

$$+\frac{jk}{4}\int_c \overline{\mathbf{M}}_s \times \hat{\mathbf{r}}\, H_1^{(2)}(kr)dc \tag{3}$$

$$\overline{\mathbf{H}}^s(\overline{\mathbf{J}}_s,\overline{\mathbf{M}}_s) = -\frac{\omega\varepsilon}{4}\int_c \overline{\mathbf{M}}_s H_0^{(2)}(kr)dC + \frac{k}{4\omega\mu}\int_c (\nabla_s \bullet \overline{\mathbf{M}}_s)\,\hat{\mathbf{r}}\, H_1^{(2)}(kr)dC$$

$$-\frac{jk}{4}\int_c \overline{\mathbf{J}}_s \times \hat{\mathbf{r}} H_1^{(2)}(kr)dC \tag{4}$$

where μ, ε are the free-space constituent parameters, and η is the free-space wave impedance. The free-space wavenumber is $k = 2\pi/\lambda$, with λ being the free-space wavelength, and ω is the angular frequency. The incident (impressed) electric and magnetic fields can be expressed as

$$\overline{\mathbf{E}}^{imp} = \hat{\mathbf{z}}\mathbf{e}^{-jkx} = \hat{\mathbf{z}}e^{jk\rho\cos(\phi-\phi^i)} = \hat{\mathbf{z}}e^{jk\rho\cos(\phi-\pi)} \tag{5}$$

$$\overline{\mathbf{H}}^{imp} = \hat{\mathbf{I}}^i \times \hat{z}\frac{1}{\eta}e^{jk\rho\cos(\phi-\phi^i)} \tag{6}$$

Note that (ρ,φ,z) is the cylindrical coordinate system , $\hat{\mathbf{I}}^i$ is a unit vector in the direction of propagation and $\phi^i = \pi$. It can easily be shown that

$$\overline{\mathbf{J}}_{s1} = \hat{\mathbf{n}}_1 \times \overline{\mathbf{H}}^{imp} = \hat{\mathbf{z}}\cos(\phi-\pi)\frac{1}{\eta}e^{jkR\cos(\phi-\pi)} = J_Z\hat{\mathbf{z}} \tag{7}$$

$$\overline{\mathbf{M}}_{s1} = \overline{\mathbf{E}}^{imp} \times \hat{\mathbf{n}}_1 = \hat{\boldsymbol{\varphi}}e^{jk\cos(\phi-\pi)} = M_\phi\hat{\boldsymbol{\varphi}} \tag{8}$$

The scattered fields at an observation point x_o,y_o are obtained from the radiation integrals and can be shown to be

$$\overline{\mathbf{E}}^s\ (J_z, M_\phi) = -\frac{kR}{4}\hat{\mathbf{z}}\int_0^{2\pi}\cos(\phi-\pi)e^{jkR\cos(\phi-\pi)}H_0^{(2)}(kr)d\phi \tag{9}$$

$$-\frac{jkR}{4}\hat{\mathbf{z}}\int_0^{2\pi}\cos(\phi-\theta)e^{jkR\cos(\phi-\pi)}H_1^{(2)}(kr)d\phi$$

$$\overline{\mathbf{H}}^s(J_z, M_\phi) = \frac{jkR}{4\eta}\int_0^{2\pi}\cos(\phi-\pi)e^{jkR\cos(\phi-\pi)}[\sin\theta\,\hat{\mathbf{x}} - \cos\theta\,\hat{\mathbf{y}}]H_1^{(2)}(kr)d\phi$$

$$-\frac{kR}{4\eta}\int_0^{2\pi}e^{jkR\cos(\phi-\pi)}[-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}]H_0^{(2)}(kr)d\phi \tag{10}$$

$$-\frac{jkR}{4\eta}\int_0^{2\pi}\sin(\phi-\pi)e^{jkR\cos(\phi-\pi)}[\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}]H_1^{(2)}(kr)d\phi$$

where

$$r = \sqrt{(x_0 - R\cos\phi)^2 + (y_0 - R\sin\phi)^2} \tag{11}$$

$$\theta = \tan^{-1}\left[\frac{y_0 - R\sin\phi}{x_0 - R\cos\phi}\right] \tag{12}$$

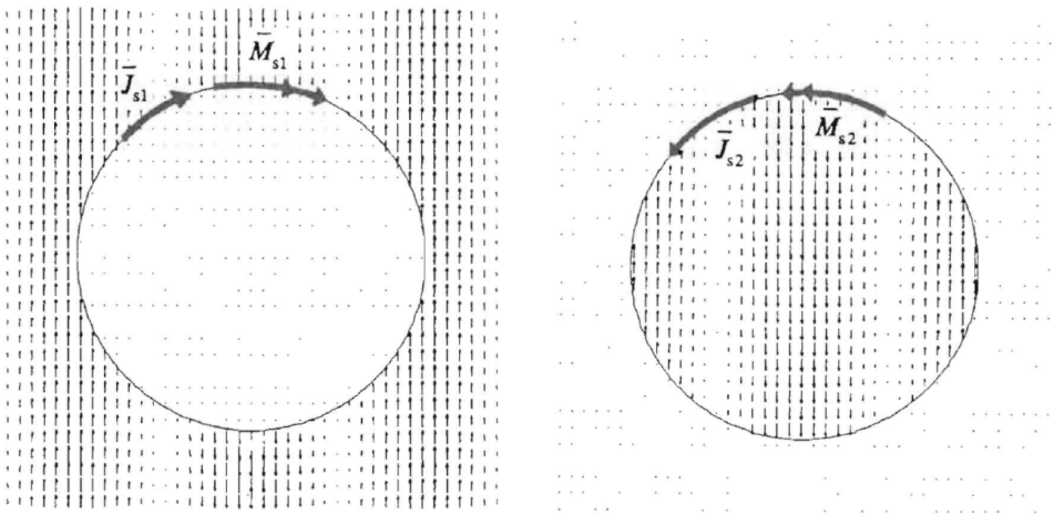
$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}} \tag{13}$$

$$\hat{\boldsymbol{\varphi}}\times\hat{\mathbf{r}} = -\hat{\mathbf{z}}\cos(\phi-\theta) \tag{14}$$

$$\hat{\mathbf{z}}\times\hat{\mathbf{r}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}} \tag{15}$$

$$\nabla_s \bullet \overline{\mathbf{M}}_s = \frac{1}{R}\frac{\partial}{\partial\phi}M_\phi = -jk\sin(\phi-\pi)e^{jkR\cos(\phi-\pi)} \tag{16}$$

Since there is no actual scatterer, these surface currents should have no contribution to the total (incident) field outside *S*, but should, together with the impressed field, maintain a null total field within *S*. This is indeed so, as can be seen in Fig. 2(a), where the real part of $\overline{\mathbf{H}}_1$ has been plotted. The total field outside *S* is equal to the impressed field, implying therefore that the scattered field must be zero. Numeric evaluation of (9) shows that the two integrals cancel at all external observation points. Similarly, the integrals in (10) also cancel at all external observation points. For internal observation points these same integrals add up to yield the negative of the impressed field, causing the total internal field to be zero. In the case of internal equivalence, $\overline{\mathbf{J}}_{s2} = -\overline{\mathbf{J}}_{s1}$ and $\overline{\mathbf{M}}_{s2} = -\overline{\mathbf{M}}_{s1}$. As there are no sources present and, hence, no impressed field, the total field is equal to the scattered field. Numeric evaluation of equations (9) and (10) yields scattered fields equal to the impressed fields inside *S*, but zero outside *S*, as shown in Fig. 2(b).



(a) External equivalence $Re\ \{\overline{\mathbf{H}}^{imp} + \overline{\mathbf{H}}^{s}(\overline{\mathbf{J}}_{s1}, \overline{\mathbf{M}}_{s1})\}$

(b) Internal equivalence: $Re\ \{\overline{\mathbf{H}}^{s}(\overline{\mathbf{J}}_{s2}, \overline{\mathbf{M}}_{s2})\}$

Figure 2. Fictitious surface currents producing $\overline{\mathbf{H}}^{imp}$

Instead of having the internal field zero, one can specify an auxiliary field $\overline{\mathbf{E}}_2^{a}$ within *S* such that $\hat{\mathbf{n}}_1 \times \overline{\mathbf{E}}_1 = \hat{\mathbf{n}}_1 \times \overline{\mathbf{E}}_2^{a}$ and hence $\overline{\mathbf{M}}_{s1} = \mathbf{0}$ [4]. For external equivalence, we have

$$\hat{\mathbf{n}} \times \overline{\mathbf{H}}^{s}(\overline{\mathbf{J}}_{s1} = \overline{\mathbf{J}}_{s}^{ext}) + \hat{\mathbf{n}} \times \overline{\mathbf{H}}^{imp} = \hat{\mathbf{n}} \times \overline{\mathbf{H}}_1 \tag{17}$$

Since $\overline{\mathbf{H}}_1 = \overline{\mathbf{H}}^{imp}$ in Equation (17), the trivial solution clearly is $\overline{\mathbf{J}}_{s}^{ext} = \mathbf{0}$, and the total field everywhere inside and outside *S* is therefore simply the impressed field.

It should be noted that the integral equation presented by (17) does not have a unique solution at resonant frequencies of the cavity formed by making the surface *S* perfectly conducting and keeping the region external to *S* free space [5]. This implies that it may be possible to have $\overline{\mathbf{J}}_{s}^{ext}$ that is non-zero, whilst still producing zero external fields.

For internal equivalence, there are no impressed fields. The total internal field should equal the incident field, and the boundary conditions on *S* would then be

$$\hat{\mathbf{n}}_2 \times \overline{\mathbf{E}}^s(\overline{\mathbf{J}}_s^{int}) = \hat{\mathbf{n}}_2 \times \overline{\mathbf{E}}^{imp} \quad (18)$$

$$\hat{\mathbf{n}}_2 \times \overline{\mathbf{H}}^s(\overline{\mathbf{J}}_s^{int}) = \hat{\mathbf{n}}_2 \times \overline{\mathbf{H}}^{imp} \quad (19)$$

Anyone of the equations (18) and (19) can be solved by means of the Method of Moments, yielding the $\overline{\mathbf{J}}_\mathbf{s}^{int}$ shown in Fig. 3. The scattered (total) magnetic field is shown in Figure 4. Note the direction of the magnetic field along the boundary *S*, which agrees with Fig. 2(a). An interesting observation is that although there is no actual shadow region, the current at the "back" of the equivalent surface tends to zero. The current shown in Figure 3 is, in fact, the exact negative of the current that would have been induced on a perfect electric conductor (PEC) of size *S*, illuminated by a plane wave. The explanation for this is quite simple. When we solve Equation (19) for a PEC, we require that the external fields be equal to the true fields and that the internal fields be zero. The latter case requires $\hat{\mathbf{n}}_1 \times \overline{\mathbf{H}}^s(\overline{\mathbf{J}}_\mathbf{s}^{ext}) = -\hat{\mathbf{n}}_1 \times \overline{\mathbf{H}}^{imp}$ within *S*. The problem presented in this paper is specified by Equation (19), and with $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$ we have $\overline{\mathbf{J}}_\mathbf{s}^{ext} = -\overline{\mathbf{J}}_\mathbf{s}^{int}$. This scenario exists in the case of interior equivalence in terms of an electric current on *S*, with both regions having the same constitutive parameters as in the original problem. An analogous situation for the planar case can be found from the examples shown in [3].

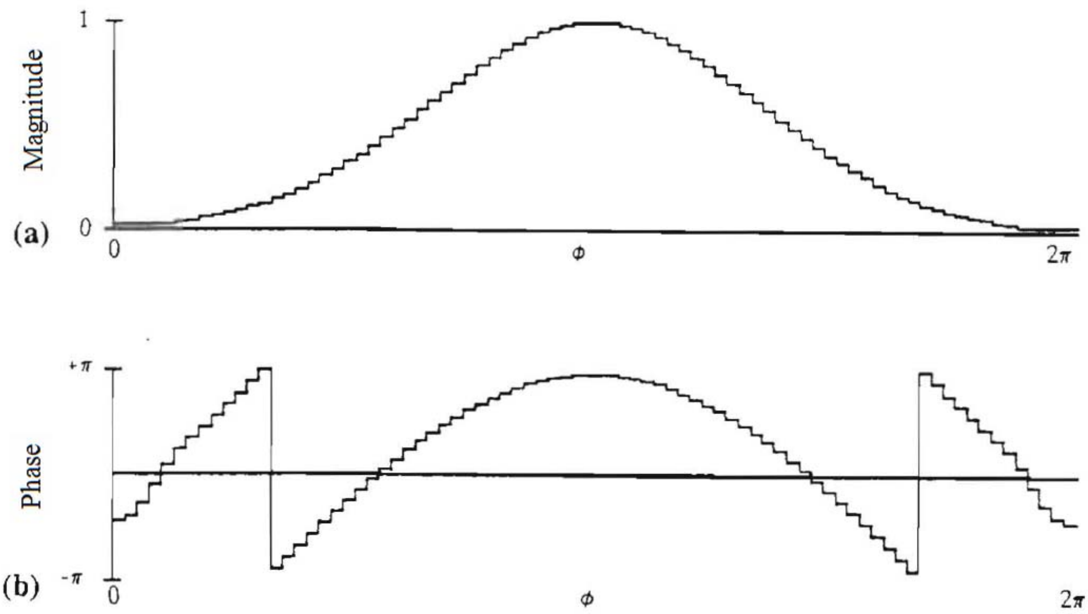


Figure 3. Equivalent $\overline{\mathbf{J}}_s^{int}$ calculated by means of the Method of Moments

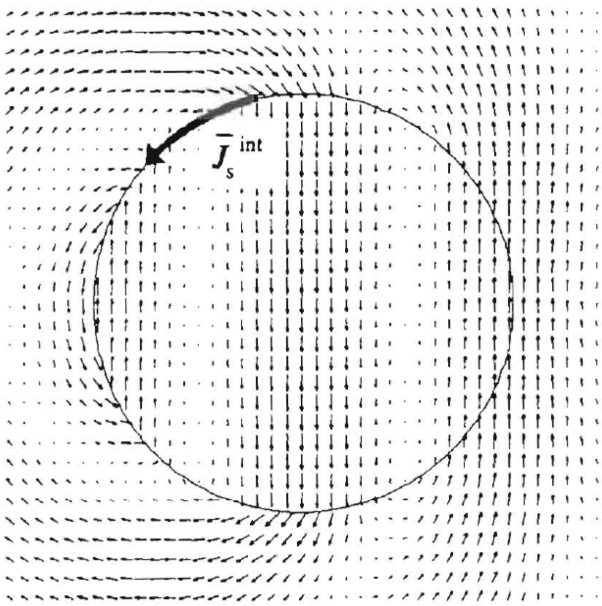


Figure 4. Total field for internal equivalence: $Re\ \{\overline{\mathbf{H}}^s(\overline{\mathbf{J}}_s^{int})\}$

The value of the problem presented lies in the interaction between the various terms in the integrals. All quantities are known analytically and can thus be calculated with ease. It is clearly shown how the true fields and null fields are generated, and how the use of an auxiliary field can limit the calculations to the use of only one of the surface current densities. The reader is referred to [6] for an example demonstrating the relationship between the equivalence theorem and the associated Geometrical Optics fields.

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